Compound Interest

Once in a while, you hear about people winning the lottery. The winnings can be several millions of dollars. The big money winners are usually paid in annual installments for about 20 years. But some of the smaller prizes are awarded in a matter of weeks. What do you think you would do if you won the lottery?

Sue's uncle gave her a lottery ticket on her 18th birthday and she won!! In the lottery payoff scheme, she has two payoff choices:

Option 1 is to receive a single $20,000 payment now.

Option 2 is to receive a single $40,000 payment in ten years.

Which option should she take?

But wait, the word is out and several banks have called to tell you about their investment plans for Option 1. One bank has offered a special 10-year certificate of deposit paying 8% interest compounded annually. Should Sue take option 1 and invest with this bank?

How do you represent and reason about functions involved in investments paying compound interest anyway?

It's not really that difficult. Let's look at each option above.

Option 2 is really easy; in 10 years from now, you'll receive 40,000.

Option 1 is not as clear. If you take option 1, you'll receive 20,000 and that is it. However, if you invest this money in the special 10 year certificate of deposit, you'll receive more. Let's find out how much more.

The basic NOW-NEXT equation would be NEXT = NOW + 0.08(NOW)

At the end of year 1, your balance will be:

20,000 + (0.08 x 20,000) = 20,000 + 1600 = $21600

At the end of year 2, your balance will be:

21600 + (0.08 x 21600) = 21600 + 1728 = $23328

At the end of year 3, your balance will be:

23328 + (0.08 x 23328) = 23328 + 1866.24 = $25194.24

Use this NOW-NEXT equation to determine how much money Sue will have in 10 years with Option 1.

$43178.50

But there has GOT to be an easier way to do this!

We know the initial value is \$20,000 and we know she's investing it for 10 years, but what is the common ratio? Calculate the value by taking a look at the final account balances Sue had for the first 3 years.

$23328 \div 21600 = 1.08$

That makes sense! Each year when Sue's interest is added, she keeps 100% of what she had the previous year and also adds 8% - so the final balance is 100% + 8% = 108% of what she had at the beginning of the year. Change 108% to a decimal and you have 1.08 for the common ratio or growth factor.

Now that you know the math behind it, we can write this as an explicit function and add in the information that we have:

$y = a \bullet (1 + r)^t$

a = initial amount

1 is there to represent 100% as a decimal

r = the % increase as a decimal

t = the time in years

Plug in the values that we have…

$y = 20{,}000(1 + 0.08)^{10}$

$y = 20{,}000(1.08)^{10}$

$y = 43{,}178.50$

Which option should Sue take? We'll I would advise her to take option 1 and to Invest the funds in the special CD. However, if she doesn't want to invest her money, then she would be better off, taking option 2. The down side here is that she will have to wait 10 years for her money. Which option would you advise Sue to take? Why?

It depends on her situation... open for discussion

YOUR TURN

Write the NOW-NEXT and explicit formulas for the following compound interest problems.

1) You have an initial investment of \$15,000 to be invested at a 6% interest rate compounded annually. What is the investment worth at the end of 5 years? What is the investment worth at the end of 15 years?

NEXT = NOW(1.06) $y = 15000(1.06)^x$ 5 years \$20073.38

start at 15000 15 years \$35948.37

2) You have an initial investment of \$7,000 to be invested at a 4.5% interest rate compounded annually. What is the investment worth at the end of 20 years? What is the investment worth at the end of 30 years?

NEXT = NOW(1.045) $y = 7000(1.045)^x$ 20 years \$16882.00

start at 7000 30 years \$26217.23

3) Sam's aunt Matilda gave him a stamp collection worth \$2,500. Sam is considering selling the collection, but his aunt told him that, if he saved it, the stamps would increase in value. Sam decided to save the collection, and its value increased by 3.75% each year. Find the value of the collection 5 years from now. When will it be worth \$5,000?

5 years \$3005.25

NEXT = NOW(1.0375) $y = 2500(1.0375)^x$

start at 2500 \$5000 in 18.82 years

More Compound Interest . . . Making even more money by investing

In the previous lesson, we explored compound interest investments where the investments were compounded annually. Many banks and credit unions offer investments which are compounded semi-annually, quarterly, monthly, or daily in order to earn even more money. We will explore some of those investments in this lesson.

Mr. Watson sold his boat for $10,000. He wants to invest the money. How much money will he have after 1 year if he invests the $10,000 in an account that pays 4% compounded interest ***per year***?

That's pretty easy now. The initial investment is $10,000, the interest rate is 4%, and the time is 1 year. Using the formula we found in the previous lesson, $y = a \bullet (1 + r)^t$, he will have:

$$Y = 10{,}000(1 + 0.04)^1$$
$$Y = 10{,}000(1.04)^1$$
$$Y = 10{,}400 \text{ at the end of 1 year.}$$

Mr. Watson sees an advertisement for another type of savings account:

4% interest ***per year compounded quarterly***

Mr. Watson doesn't really understand what "compounded quarterly" means in terms of investing, so he asks the bank officer to explain this to him. The teller explains that instead of giving him 4% of 10,000 at the end of one year, the bank will give him 1% of 10,000 at the end of each 3-month time period, which is one quarter of the year. Here's how this breaks down:

The basic NOW-NEXT equation would be NEXT = NOW + 0.01(NOW)

The initial investment is $10,000 when the year begins (0 months into the year).

At the end of 3 months, your balance will be:

10,000 + (0.01 x 10,000) = 10,000 + 100 = $10,100

At the end of 6 months, your balance will be:

10100 + (0.01 x 10100) = 10100 + 101 = $10,201

At the end of 9 months, your balance will be:

10201 + (0.01 x 10201) = 10201 + 102.01 = $10,303.01

At the end of 12 months, your balance will be:

10301.01 + (0.01 x 10301.01) = 10301.01 + 103.03 = $10,406.04

Does Mr. Watson make more money by compounding his interest once at the end of the year or every quarter? How much more money?

He makes $6.04 more when compounded quarterly instead of annually.

NOW-NEXT isn't too bad if you're only trying to see what will be in your account a year from now, but what about 12 years from now? There are 48 quarters in 12 years...that's a lot of multiplying and adding! So let's put this into an explicit formula so that we can use it in a more efficient way.

Remember, when we compound the interest once a year we used the formula $y = a \bullet (1 + r)^t$... let's tweak it for this situation. Remember, we need to divide the interest rate by 4 AND repeat this process 4 times in each year!

$$y = a \bullet \left(1 + \frac{r}{4}\right)^{4 \cdot t}$$

Why are the 4's there?? Because we have to divide our interest rate into 4 equal parts and then calculate the interest each quarter in the year.

Well, that's pretty cool...but what if the interest is compounded semi-annually (twice a year), or monthly (12 times a year) or weekly, or daily??

That's a good question...Just like quarterly though, we have to divide up the interest rate so that it's spread evenly between each time we compound the interest AND we have to compound the interest more times in the year...so let's just call that number "n"

$$y = a \bullet \left(1 + \frac{r}{n}\right)^{n \cdot t}$$

Why are the n's there?? Because we have to divide our interest rate into n equal parts and then calculate the interest n times in the year.

a = initial amount
r = interest rate
t = time in years
n = number of times the investment is compounded during one year

Back to the Story

Mr. Watson sees an advertisement for a different bank that offers 4% interest compounded monthly, which means that he will get $^1/_{12}$ of 4% interest every month. How much money will he have at the end of the year if he invests his money at this bank?

Let's try the explicit formula for this situation.

a = 10000
r = .04
t = 1
n = 12

$$y = a \bullet \left(1 + \frac{r}{n}\right)^{n \cdot t}$$ ←SUBSTITUTE!!

$$y = 10000\left(1 + \frac{.04}{12}\right)^{12 \cdot 1}$$

$$y = 14763.99$$

If you were Mr. Watson, would you suggest invest the money in an account with interest compounded annually, semiannually, quarterly, monthly, weekly, or daily? Why?

It looks like the more times the interest is compounded the more money is made, so daily would be best.

Let's do some investigations to see how to do make our investments!

YOUR TURN

The scenario is that you have $5000 to invest and you want to know which of the following investment situations will give you the most money at the end of 5 years. The interest rate for all of the situations is 6%. * Make sure you put your exponents in parentheses.

1. Calculate the investment if it is compounded annually.

$$y = 5000(1+.06)^5 = \$6691.13$$

2. Calculate the investment if it is compounded semi-annually (twice a year).

$$y = 5000\left(1+\frac{.06}{2}\right)^{5\cdot 2} = \$6719.58$$

3. Calculate the investment if it is compounded quarterly (four times a year).

$$y = 5000\left(1+\frac{.06}{4}\right)^{5\cdot 4} = \$6734.28$$

4. Calculate the investment if it is compounded monthly (12 times a year).

$$y = 5000\left(1+\frac{.06}{12}\right)^{5\cdot 12} = \$6744.25$$

5. Calculate the investment if it is compounded daily (365 times a year).

$$y = 5000\left(1+\frac{.06}{365}\right)^{5\cdot 365} = \$6749.13$$

6. What did you discover? Which situation will give you the most? Which situation is the most realistic for banks? Which situation is the most realistic for credit card companies? Explain your reasoning.

The more often you compound the interest, the more money you make!
365 times a year gives you the most money
Most banks/credit card companies add interest once a month when statements come out.

Independent Practice with Compound Interest

Write a NOW-NEXT and explicit equation for each problem situation in order to find the solution.

1) An investment of $75,000 increases at a rate of 12.5% per year. Find the value of the investment after 30 years. How much more would you have if the interest is compounded quarterly? $442931.67 more

NEXT = NOW • 1.125

Start @ 75000

$y = 75000(1.125)^x$ ← annually $2,568,247.87

$y = 75000\left(1 + \frac{.125}{4}\right)^{30 \cdot 4}$ quarterly $3,011,179.54

2) Suppose you invest $5000 at an annual interest of 7%, compounded semi-annually. How much will you have in the account after 10 years? Determine how much more you would have if the interest were compounded monthly. $1490.44 more

NEXT = NOW · 1.035

Start @ 5000

$y = 75000\left(1 + \frac{.07}{2}\right)^{x \cdot 2}$ Semi-annually $149234.16

$y = 75000\left(1 + \frac{.07}{12}\right)^{x \cdot 12}$ monthly $150724.60

3) Lisa invested $1000 into an account that pays 4% interest compounded monthly. If this account is for her newborn, how much will the account be worth on his 21st birthday, which is exactly 21 years from now?

NEXT = NOW • $1.00\overline{3}$

Start @ 1000

$y = 1000\left(1 + \frac{.04}{12}\right)^{21 \cdot 12}$ = $2313.13

4) Mr. Jackson wants to open up a savings account. He has looked at two different banks. Bank 1 is offering a rate of 5.5% compounded quarterly. Bank 2 is offering an account that has a rate of 8%, but is only compounded semi-annually. Mr. Jackson puts $6,000 in an account and wants to take it out for his retirement in 10 years. Which bank will give him the most money back?

BANK 1: NEXT = NOW(1.01375), Start @ 6000

$y = 6000\left(1 + \frac{.055}{4}\right)^{4 \cdot 10}$ $10360.62

BANK 2: NEXT = NOW · 1.04, Start @ 6000

$y = 6000\left(1 + \frac{.08}{2}\right)^{2 \cdot 10}$ $13146.74

5) Mason deposited $2,000 into a savings account that pay an annual interest rate of 9% compounded annually. Determine the amount of money in the savings account after 1 year, 5 years, 10 years and 20 years. Using the calculated values, construct a graph.

$y = 2000(1.09)^x$

1 year	$2180
5 years	$3077.25
10 years	$4734.73
20 years	$11208.82